A Criterion of Naturalness in Renormalization

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The sensitivity criterion is widely used in measuring the level of fine-tuning, although many examples show it doesn't work under certain circumstances. We discuss the mathematics behind the fine-tuning problems, explain the mathematical meanings of the sensitivity criterion, point out three implicit assumptions behind this criterion. Because of these assumptions, the sensitivity criterion can't reflect the fine-tuning level correctly. By analyzing two well known examples that the sensitivity criterion failed, we point out the dimensional effect is the main reason why we have these problems. To solve these problems, we proposed a new criterion to replace the sensitivity criterion.

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I. INTRODUCTION: THE NATURALNESS PROBLEM

The principle of naturalness introduced by Wilson and 't Hooft[1] requires that in order to obtain a weak scale observable parameter of the order M_W, we do not need to extremely fine-tune the fundamental Lagrangian parameters at the grand unification scale. Although we have different fine-tuning problems in different models, generally, these problems can be categorized into two types: The first type fine-tuning problems, which existed in renormalization. The second type fine-tuning problems, which existed in the mixing mechanisms or when parameters are linked together by matrix transformation.

The first type fine-tuning problems, for example, the renormalization of ϕ^4 model:

$$\mathcal{L} = \frac{1}{2}[(\partial_\mu \phi)^2 - m_0^2 \phi^2] - \frac{g}{4!}\phi^4 \tag{1}$$

the renormalized scalar mass m^2 is:

$$m^2 = m_0^2 - g^2 \Lambda^2 \tag{2}$$

where m_0 is the bare mass, Λ is the cut-off energy scale. Because both m_0 and Λ are around 10^{18} GeV, thus to obtain a small weak scale renormalized mass m, we need a fine-tuning mechanism to match the values of m_0 and Λ.

For the first type fine-tuning problems, generally, the parameter range at the weak scale(which is 0 to M_W roughly) is much smaller than the corresponding parameter range at the grand unification scale (which is 0 to M_P roughly). The parameter range shrinks rapidly as the energy scale decreases. Here the fine-tuning problem is due to the large parameter range shrinkage from the grand unification scale to the weak scale (This will be more clearly if it is expressed in a renormalization group equation). It happens when we compare parameters with different energy scales, and once we given the value of the weak scale parameters, the large parameter range at the grand unification scale will be controlled by the renormalization group equation itself.

The second type fine-tuning problems generally is not related with renormalization, they are related with the parameter transformation at the same energy scale, for example, by mass mixing. Take the MSSM tree level Z-boson mass m_Z as an example:

$$\frac{1}{2}M_Z^2 = \frac{M_{H_D}^2 - M_{H_U}^2 \tan^2\beta}{\tan^2\beta - 1} - \mu^2 \tag{3}$$

Because both M_{H_D}, M_{H_U} and μ could be as large as 10^5 GeV [2] , while M_Z is only around 100 GeV, again we need a fine-tuning mechanism to match the values of M_{H_U} and μ. Besides this example, We will find this types of fine-tuning problems in the various mass mixing mechanisms in the MSSM model.

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The example of the MSSM model Z-boson mass fine-tuning problem shows, for this type fine-tuning problems, the relation between the input parameters ($M^2_{H_D}$, $M^2_{H_U}$ and μ^2) and the output parameter (M^2_Z) is linear (or not far away from linear for other similar examples in mass mixing mechanisms), which means not like in the first type where the parameter range shrinks 10^{18} times, the possible range of the output parameter M^2_Z is almost the same as the possible range of the input parameters $M^2_{H_D}$, $M^2_{H_U}$ and μ^2. Thus the fine-tuning is mostly due to the possibility is extremely low to pick up a specific parameter value in an extremely large parameter range. For this type, the large parameter range is not decided by the mixing mechanism or the transformation itself, it is given by other mechanisms.

These two types of fine-tuning problems have different properties, we may need different methods to describe them.

II. THE SENSITIVITY CRITERION AND ITS PROBLEMS

The sensitivity criterion proposed by R. Barbirei and G.F.Giudice et al.[3] is the first widely adopted criterion in this field, it uses the sensitivity parameter to quantitatively measure the naturalness level. If x is a fundamental Lagrangian parameter at the grand unification scale, and y is a computed observable parameter such as weak scale masses, Yukawa couplings. If we varies the Lagrangian parameter x at the grand unification scale, based on the corresponding variation of the weak scale observable parameter y, the sensitivity parameter c is defined as:

$$c(x_0) = \left| \frac{x}{y} \frac{\partial y}{\partial x} \right|_{x=x_0} \tag{4}$$

here need to emphasis that the Lagrangian parameter x and the observable parameter y may have different units, while the sensitivity parameter c will be used to compare the fine-tuning properties of different models. We would like to see it in a dimensionless formation. So in the definition of the sensitivity parameter c, relative variations $\delta y/y$ and $\delta x/x$ are used as the basis of the comparison.

Barbieri and Giudice chose $c = 10$ as the maximum allowed sensitivity for any models to be categorized as "natural", if $c \gg 10$ then it is "unnatural" (or fine-tuned). The sensitivity criterion has been widely adopted since then. It has been used in many fields, for example, setting a naturalness contour for SUSY particle search[4], or the fine-tuning problem of the neutrino seesaw mechanism[5]. But people soon found it is not a reliable criterion. Many examples show that sometimes the sensitivity criterion fails. The most famous examples among them are given by G. Anderson et al[6] and P. Ciafaloni et al[9].

The example given by G. Anderson et al[6] is regarding the high sensitivity of Λ_{QCD} scale to the variation of the strong coupling constant g. Because the relation between Λ_{QCD} and the strong coupling constant g is:

$$\Lambda_{QCD} = M_P \exp\left(-\frac{(4\pi)^2}{b g^2(M_P)}\right) \tag{5}$$

the corresponding sensitivity parameter c is:

$$c(g) = \frac{4\pi}{b} \frac{1}{\alpha_s(M_P)} \gtrsim 100 \tag{6}$$

according to the sensitivity criterion, it is fine-tuned. But we know actually Λ_{QCD} is protected by the gauge symmetry, it is not fine tuned.

The example given by P. Ciafaloni et al[9] is about the high sensitivity of the Z-boson mass. When the Z-boson mass M_Z is dynamically determined through gaugino condensation in a "hidden" sector, the mass M_Z can be expressed as:

$$M_Z \approx M_P e^{-l/g_H^2} \tag{7}$$

Where g_H is the hidden sector gauge coupling constant renormalized at M_p, and l is a constant. Like the first example, the calculated sensitivity c for the second example is also much larger than 10, the maximum allowed sensitivity, although we know Z-boson mass is also not fine-tuned.

Compare Eq. (5) and Eq. (7), both of them have a similar mathematical formation. Actually, if we calculate the sensitivity c of any weak scale mass by varying a related coupling constant at the grand unification scale, the calculated sensitivity is always very high no matter whether the mass is really fine-tuned or not. These examples show the sensitivity criterion is not reliable. Although this problem has been pointed out long time ago, yet we still can find people use the grand unification scale coupling constant to measure the level of the fine-tuning[5].

Besides these well known problems, the sensitivity criterion also has other problems. If we apply the sensitivity criterion (Eq. (4)) to the scalar mass of ϕ^4 model (Eq. (1)), clearly from Eq. (2) we know the scalar mass is highly fine-tuned. But on the other hand, because both m_0^2 and $g^2\Lambda^2$ in Eq. (1) are not independent, so if we integrate the mass renormalization group equation, we have:

$$m^2 = m_0^2 e^{\int_0^t (g^2/16\pi^2 - 1)dt} \tag{8}$$

Thus the sensitivity $|\partial \ln m^2/\partial \ln m_0^2|$ equals to one, and it is not fine-tuned.

Obviously, the origin of this problem is the ratio $\delta m^2/m^2$ is not depended on the energy scale. Thus even δm^2 is only around hundreds GeV^2 while m^2 is around $10^{36}\mathrm{GeV}^2$(which means highly fine-tuned), the sensitivity is still one. This reminds us the sensitivity c can not reflect the level of the fine-tuning when it is due to the shrinkage of the parameter range.

Generally, for renormalization, if we integrate a renormalization group equation, roughly speaking, an observable parameter y can be expressed as a function of the energy scale:

$$y = y_0 e^{\int (n - \gamma(y_0))dt} \tag{9}$$

where n is the naive dimension and γ is the anomalous dimension. If the naive dimension n is not equal to zero, for example, ϕ^4 model scalar mass, then there will be a fine-tuning problem. If we calculate the sensitivity c of the weak scale parameter y when we varies y_0 at the grand unification scale, the corresponding sensitivity c is:

$$c(y_0) = 1 - \frac{\partial}{\partial \ln y_0} \int \gamma(y_0)dt \tag{10}$$

Which means the sensitivity c can only reflect the contribution from the anomalous dimension, and will ignore the major contribution from the big naive dimension. So in some sense it is more accurate to call the sensitivity c as the anomalous sensitivity when dealing with the first type fine-tuning problems.

Based on the above analysis, we learn that the sensitivity criterion has many problems. First it can not be applied to the first type fine-tuning problems, and the second, it is incorrect when dealing with parameters with different dimensions. To solve these problems, many authors proposed alternative prescriptions. G. Anderson et al [6, 7, 8]. first introduced the idea of probability distribution, they argued that, the relative variation could have a probability distribution, we need to use $\bar{c}$, the probability average of the sensitivity c, and the sensitivity criterion should be replaced by:

$$\gamma = c/\bar{c} \tag{11}$$

Based on this criterion, only those with $\gamma \gg 1$ should be considered as fine-tuned.

Later another modified definition was proposed to replace the original sensitivity criterion [9, 10, 11, 12, 13]:

$$c(x_0) = \left| \frac{\Delta x}{y} \frac{\partial y}{\partial x} \right|_{x=x_0} \tag{12}$$

where Δy is the experimentally allowed range of the parameter x.

Although many authors attempted to give a correct numerical description of the naturalness level, none of them can claim quantitative rigor. The calculated naturalness level usually depends on what criterion we used, and these results may reflect the naturalness properties correctly or incorrectly. Because the sensitivity criterion plays a very important role, it is worth to investigate the relationship between the naturalness and the sensitivity, thus we can find a correct and reliable criterion.

III. MATHEMATICAL BASIS OF THE NATURALNESS CRITERION

The naturalness problem is similar to the initial condition sensitivity problem in mathematics, which means small differences in initial conditions will result in big differences as the system evolves. The final state of a system will extremely depend on small variations of the initial condition. In mathematics, sensitivity to initial conditions can be measured quantitatively in many ways, for example, by the Lyapunov exponent. These methods are all based on the probability theory, which reflects how large the probability is to find an output parameter to be in a certain parameter range.

According to the probability theory, if parameters x and y are linked by $y = f(x)$, and if $p[x]$ is the original probability distribution function of the parameter x, and if the function $f(x)$ is monotonic, then the probability distribution of the parameter y will be [14]:

$$p(y) = p[f^{-1}(y)]\left|\frac{\partial}{\partial y}f^{-1}(y)\right|_{x=x_0} \tag{13}$$

Note the probability distribution $p(y)$ is controlled by the original distribution of the parameter x (which is $p[f^{-1}(y)]$) and the transformation between the parameters x and y (which is $|\partial x/\partial y|$).

By means of the probability theory, we can interpret the sensitivity criterion more clearly. If we calculate the sensitivity of a parameter y by varying x around $x = x_0$, if we assume that the input parameter x is uniformly distributed from 0 to x_0 (probability density $p[f^{-1}(y)] = 1/x_0$), then the total probability of the output parameter y to be found in the region $0 \leq y \leq y_0$ is:

$$P = \int_0^{y_0} \frac{y_0}{x_0}\left|\frac{\partial x}{\partial y}\right| dy \approx \frac{y_0}{x_0}\left|\frac{\partial x}{\partial y}\right|_{x=x_0} \tag{14}$$

Where P represents the total probability of the parameter y to be found within $0 \leq y \leq y_0$. In the second part of Eq. (14) it is assumed that $\partial x/\partial y$ is a constant in the region $0 \leq y \leq y_0$. Compare Eq. (14) with Eq. (4), we found that, if the total probability of y to be found in the region $0 \leq y \leq y_0$ is P, then the sensitivity parameter c is approximately the inverse of this probability:

$$c = \frac{1}{P} \tag{15}$$

If we choose $c_{\max} = 10$ as the maximum allowed sensitivity, by means of the probability theory, it means the minimum allowed total probability of the observable parameter y to be found within the range $[0, y_0]$ is approximately 10%.

Based on the above analysis, it is not difficult to see there are three assumptions for the sensitivity criterion:

1. The sensitivity criterion implies the Lagrangian parameter x is evenly distributed in $0 \leq x \leq x_0$ only. The probability distribution in the region $x > 0$ is zero.

2. What the sensitivity criterion calculated is approximately the *total probability* of the observable parameter y to be found in the region $0 \leq y \leq y_0$.

3. The sensitivity criterion also implies that the relation between the grand unification scale parameter and the weak scale parameter is monotonic.

The first two assumptions are the main reasons why the sensitivity criterion fails in the case of renormalization. For the first assumption, because for a fine-tuning problem associated with renormalization, it is usually hard to define an upper bound of an input parameter naturally. The sensitivity criterion implies the value x_0 is the upper bound of the input parameter x, and ignore the possibility of $x \geq x_0$. Obviously, P. Ciafaloni did aware this problem, and change the definition of the sensitivity from its original version (Eq. (4)) to the modified one(Eq. (12)), which means it is assumed that the Lagrangian parameter x uniformly distributed from 0 to the experimentally allowed maximum value Δx. Certainly this is a step ahead toward the correct prescription but we also have difficulties to choose a specific experimentally allowed maximum value.

The second assumption means that what the sensitivity c tells us is always the average fine-tuning level of an observable parameter y to be found in the region $0 \leq y \leq y_0$. It does not reflect the exact fine-tuning level at $y = y_0$. This problem will be even worse in renormalization. Because for the fine-tuning problems related with renormalization, parameter ranges $[0, y_0]$ and $[0, x_0]$ are related mathematically. According to our previous discussion, the sensitivity parameter can only reflect the contribution from the anomalous dimension. It will give a wrong answer, although the big ratio of y_0/x_0 is the origin of the fine-tuning problem. These assumptions especially the second one will have severe consequences under some special conditions.

For the third assumption, because Eq. (13) is only correct when $y = f(x)$ is monotonic, thus the sensitivity criterion also implied the relation between the input and the output is monotonic, which means we can't find an example that both input parameters x_1 and x_2 will eventually lead to a same output parameter y, certainly, it is not true. Because although the relationships between parameters linked by most renormalization group equations are monotonic, while most mixing cases are not. One output parameter usually corresponds to two input parameters. For example, mixing of M_Z mass and M_W mass, mixing of CP-even Higgs masses in Supersymmetric Standard Models, and mixing of fermionic masses etc.

Take the mixing of M_Z mass in MSSM model as an example, calculate the M_Z mass at the initial condition (at grand unification scale) $m = 200$ GeV, $M = 40$ GeV, $\tan\beta = 18$, and gradually reduced mass m, then we will find the non-monotonic relationship between grand unification scale variable m and weak scale variable M_Z.

Obviously, M_Z is not a monotonic function of m, for example, if weak scale mass M_Z is around $80GeV$, there are two grand unification scale parameter regions that can contribute this result, one is around m is 80 GeV, the other region is around m is 150 GeV. Barbieri and Giudice's definition only counted one region's contribution , it will overestimate the naturalness level. We should count all possible GUT scale parameters contributions.

IV. DIMENSIONAL EFFECT

It is meaningless to compare two different parameters with completely different units. In order to do so we need first convert them to a comparable format. The sensitivity criterion uses the relative variation to convert a dimensionful parameter y to a dimensionless format $\delta y/y$. It looks like the problem of comparing parameters with different units has been solved, but the examples given out by G. Anderson et al[6](Eq. (5)), and P. Ciafaloni et al[9](Eq. (7)) remind us it still has problems when comparing parameters with different units.

Generally, if we only consider the contribution from the naive dimensions and ignore the small corrections like the anomalous dimensions, physical parameters can be categorized into two types: scale invariance conserved and the scale invariance broken[15]. The scalar mass of ϕ^4 model is an example of scale invariance conserved. As the energy scale Λ increases, the value of the parameter will also be increased as an exponential function Λ^α (where α is the naive dimension); While the fermion masses is an example of the scale invariance broken, where the value of the parameter will not increase as an exponential function of the energy scale Λ. We can find that if the naive dimension of a parameter is not equal to zero, those parameters with scale invariance conserved will have the fine-tuning problem in renormalization, because the values of these parameters will blow up quickly as the energy scale increases.

If the scale invariance is conserved, for two parameters τ and h with different naive dimensions, In renormalization, if we change the energy scale Λ, then these two parameters τ and h will have the following relations:

$$\tau \approx \Lambda^\alpha \tilde{\tau} \tag{16}$$

$$h \approx \Lambda^\beta \tilde{h} \tag{17}$$

where α and β are the corresponding naive dimensions. Because of the renormalization, the energy scale Λ will connect these two parameters together, even they may not have any other mathematical relation. If we calculate the sensitivity of h to the variation of τ, because:

$$\frac{\partial h}{\partial \tau} \approx -\frac{\beta}{\alpha}\frac{h}{\tau} \tag{18}$$

we can immediately calculate the sensitivity c is β/α. This sensitivity is a consequence of comparing two parameters with the different naive dimension. If both τ and h has the same naive dimension, then the sensitivity will be one, thus the dimensional effect will disappear. But when they have different canonical dimensions, this factor will not be one, it will greatly affect the sensitivity.

This effect is known as scaling effect in statistical physics, which exists anywhere when two parameters have different dimensions. Obviously, it has nothing to do with the fine-tuning. In the definition of the sensitivity parameter c, in order to compare parameters with different dimensions, the relative variation $\delta y/y$ was introduced to remove the effects of different dimensions, but based on Eq. (18), we find the dimensional effect is a nonlinear effect, can not be eliminated by the relative variation $\delta y/y$. When considering the fine-tuning problems in renormalization, the effect of different naive dimension $-\beta/\alpha$ is still there. This is the reason why the sensitivity criterion failed in the examples given by G.W.Anderson et. al. and P.Ciafaloni et. al.

If one parameter τ has a marginal naive dimension, or when the scale invariance is not conserved, then we can not use the above argument, we must consider the higher order term. take the coupling constant renormalization as an example:

$$\tau \approx (1 + \alpha\tilde{\tau}\ln\frac{1}{\Lambda})^\delta \tilde{\tau} \tag{19}$$

where δ will be 1 for parameters with zero naive dimension (for example, the gauge coupling constants), or a non-zero number (for example, the fermion masses). Corresponds to Eq. (18), we have:

$$\frac{\partial h}{\partial \tau} \approx \frac{\beta}{\alpha\tau\delta}\frac{h}{\tau} \tag{20}$$

The corresponding sensitivity c will be $\beta/\alpha\tau\delta$. Obviously, this factor can be very large. For example, the problem of the Z boson mass discussed previously, even it is a fermion with zero dimension, we still find large sensitivity when comparing with the coupling constant, this is due to the factor δ here is not zero. Once again, this reminds us that the dimensional effect is a nonlinear effect, the relative variation $\delta y/y$ used in the sensitivity criterion can only remove linear effect. Thus even the unit has been removed by using $\delta y/y$, its effect is still there. The factor $\beta/\alpha\tau\delta$ is the contribution of the dimensional effect to the sensitivity.

For convenience we define a dimensional effect factor: $\Delta = \beta h/\alpha\tau$ or $\beta h/\alpha\delta\tau^2$ for later reference.

V. DEFINITION OF THE NEW CRITERION

As we discussed in the previous sections, the traditional sensitivity criterion fails mainly for two reasons, first, it uses the total probability instead of the probability density, which can only reflects the average sensitivity of the whole parameter range. For the fine-tuning problems in renormalization, because the total probability is not greatly changed even when the parameter is fine-tuned, using the total probability will give the wrong result. Second, because the sensitivity criterion does not include the contribution from the dimensional effect, so it fails when comparing parameters with different dimensions.

Generally, there are two types of the fine-tuning problems, either because of the parameter space greatly shrunk from the input to the output (in renormalization), or because of the parameter range for both input and output are extremely large(in mass mixing).

For the first type, as we have discussed, it is not a good idea to use the total probability to calculate the fine-tuning level, because a large naive dimension is the main reason why we have the fine-tuning problems in renormalization, while the constant part of the dimension will not change the total probability, and this part happens to be the main reason of the fine-tuning. So we need a criterion that can count the contribution from the whole naive dimension. Besides, for this type we also need to consider the dimensional effect when comparing parameters with different mass dimensions.

While for the second type, because it happens at the same energy scale. For the sensitivity criterion, we only need to face the problem of using the total probability. Using the total probability in some sense does reflect the fine-tuning level qualitatively, although strictly speaking it only gives an average level of fine-tuning in a certain range. So it is more important to find a new fine-tuning criterion for the first type.

Before proceed to propose a new prescription, we have to solve two problems. First, because we don't know the distribution function of the Lagrangian parameters at the grand unification scale. Without any experimental evidence, we have to assume that these Lagrangian parameters are all evenly distributed.

Second, how to choose the maximum allowed values of these parameters is another important problem. because this will also greatly affect the final result. In the sensitivity criterion, it is implied that the parameter value we want to measure the fine-tuning level is the maximum allowed value. Later P. Ciafaloni et al. modified it to the experimentally allowed range[9, 10, 11, 12, 13]. Certainly, for the fine-tuning problems existed in mixings (second types), because the parameter range is not specified by the mixing mechanism itself, so we can choose a maximum allowed value (either experimentally or theoretically) as the maximum parameter range. While for the first type, unlike the second type, the maximum allowed parameter range is usually given by the renormalization itself. Once the weak scale is given, the maximum parameter value at the grand unification scale will be controlled by the naive dimension. For example, for ϕ^4 model, we need to assume at the grand unification scale, the maximum allowed mass is around 10^{18} GeV, while for a fermion mass, we only need to assume the maximum allowed mass is around couple hundred GeV. Now if we want to compare a ϕ^4 model scalar mass with a fermion mass, which maximum allowed mass we should choose? 10^{18} GeV? or couple hundred GeV? Either way will be a disaster for any fine-tuning criterion.

In mathematics, sensitivity to initial conditions can be measured quantitatively in different ways, for example, by Lyapunov exponent [16], which measures the separation of two neighboring trajectories in the phase space. Using the language of the probability theory, it corresponds to measure the variation of the probability distribution. This problem is almost identical to the fine-tuning problems existed in renormalization, we can borrow its idea to define a new fine-tuning criterion in renormalization.

In order to solve the problems we discussed previously, we can not compare the relative variation $\delta x_0/\delta x$ with $\delta y_0/\delta y$, instead, we need to use $\delta y/\delta x$ directly. Here the factor $\delta y/\delta x$ means the ratio between the variation δx and the corresponding variation δy. Just like the problem of sensitive to initial conditions in mathematics, the Lyapunov exponent (corresponds to the dimension in physics) decides the final result, we don't need to give a maximum allowed value. For the fine-tuning problems in renormalization, use $\delta y/\delta x$ directly, will solve all the problems in the sensitivity criterion.

Certainly, we need to consider the dimensional effect, if two parameters x and y doesn't have any other mathematical relation except linked by the energy scale Λ in renormalization, because of the dimensional effect, $\partial y/\partial x$ will be equal

to the dimensional factor Δ we defined earlier. If both parameters have the same naive dimension, then the factor Δ becomes one. This factor is a kind of background probability density which should always be subtracted when consider the fine-tuning problems. After subtracted the factor Δ, the remaining part of $\delta y/\delta x$ is due to the fine-tuning mechanism.

Because both $\delta y_0/\delta y$ and the dimensional factor Δ has the same mass dimension, in order to have a dimensionless fine-tuning measurement, refer to the definition of the Lyapunov exponent, if we take the dimensional effect factor Δ_0 as the initial separation of two neighboring renormalization group orbits, then the separation between these two orbit can write as the function of t:

$$\frac{\delta y}{\delta x} = \Delta_0 e^{\lambda} \tag{21}$$

Where Δ_0 is the dimensional factor at the grand unification scale. Thus the factor λ here is always dimensionless.

According to Eq. (14), $\partial y/\partial x$ is identical to the probability density, the dimensionless factor λ_0 at the grand unification scale reflects the background probability density. So the factor λ represents how difficult to choose a specific value at the weak scale. Using the language of the probability theory, Eq. (21) means after considered the dimensional effect, if at the weak scale one parameter has a fluctuation in a small area, then at the grand unification scale, the probability of the corresponding parameter still within this small area is $\exp\lambda$. Because we didn't use the relative variation in the beginning, so clearly, this prescription won't have any problems the sensitivity criterion has.

Just like the Lyapunov exponent, the factor λ here corresponds to the naive dimension. If a parameter has a big naive dimension, for example, the scalar mass of ϕ^4 model, we will have a large positive λ, which means as the energy scale increase, the scalar mass increases rapidly and will have a fine-tuning problem. While for a fermion masse, usually, we will have a small negative λ, which means as the energy scale increases, the fermion masses decrease slowly thus won't have the fine-tuning problem.

Now if considering the non-monotonic propriety, according to the probability theory, we should divide the whole parameter region into several monotonic sections, and sum up the corresponding probability densities. Based on this, we can modify the previous expression to the following form:

$$\lambda = \ln \sum \left| \frac{1}{\Delta_0} \frac{\delta y}{\delta x} \right| \tag{22}$$

Which means if it is not monotonic, we divide and sum up the exponent over all monotonic regions.

VI. DEFINITION THE MAXIMUM TOLERATED λ

Although the problem of the fine-tuning criteria in high energy physics is somewhat similar to the problem of using Lyapunov exponent to judge whether a nonlinear system is sensitive to initial conditions or not, these two situations also have important differences.

In nonlinear physics, if the Lyapunov exponent is negative, then the phase space of a system shrinks as time increases, it is obviously not initial condition sensitive. If the Lyapunov exponent is positive, then the system is initial condition sensitive and thought it is chaos. Similarly, when we consider the fine-tuning problem in high energy physics, if $\lambda < 0$, for the same reason, we can easily classify the system as not initial condition sensitive or not fine-tuned. But for the cases with $\lambda > 0$, the situation is a little more complex. This is because for systems in nonlinear physics, the time variable t can go to infinity while in high energy physics, the running parameter t can not go beyond the grand unification scale, which is around 38. So for the situations with small positive λ, even the corresponding variation in the weak scale is a little less than the variation in the grand unification scale, it still can be thought as not initial condition sensitive, or not fine-tuned. So a small positive λ should not be considered as fine-tuned.

In probability theory people usually define the probability $P \leq 0.05$ as the limit of small probability events, although this is a more strict condition than the sensitivity less than 10 criterion (which means the $P \leq 0.1$), because the criterion of $P \leq 0.05$ is widely used in probability theory, so here it is used to define an upper limit for the fine-tuning level. Suppose fine-tuning occurs when $P \leq 0.05$, or $\exp(\lambda) \leq 1/0.05$. we immediately have the upper limit of the λ is 3. According to this definition, all parameters with $\lambda \leq 3$ will be safe and not fine-tuned, and if $\lambda > 3$ which means it is less than 5% of chance to have this weak scale value thus it is quite impossible. Not like Barbieri and Giudice's naturalness $c \approx 10$ cut-off, which doesn't have any physical meaning, our method gives a clear physical meaning of the naturalness cut-off.

VII. EXAMPLES

1. The Λ_{QCD} problem: The sensitivity criterion gives $c \gtrsim 100$, it is fine-tuned, which is wrong. For our new criterion, it is not difficult to calculate that $\lambda = (\ln \frac{\Lambda_{QCD}}{gM_P})$, which is far less than 3. It is not fine-tuned,

2. For non-monotonic problem:

   for non-monotonic case M_z, if $M_z = 89.05$ GeV, which corresponds to $m = 188.5$ GeV and $m = 65.1$ GeV at the grand unification scale, we calculated that sensitivity $c = 0.499$ and $c = 0.910$ respectively. while λ for $M_z = 89.05$ GeV is -1.748. Although both of them don't have much difference, while if we use the fine-tuning level to set the allowed parameter range to guide our new particle search, then the contour will be wrong.

3. Φ^4 model scalar mass problem:

   for Φ^4 model scalar mass, which has engineering dimension equals to 1, sensitivity $c = 1 < 10$, which is wrong. While our new criterion $\lambda = 1 + \frac{g^2_{GUT}}{32\pi^2}$, greater than 3.

All these examples show that our new naturalness criterion can solve all problems the sensitivity criterion has.

VIII. CONCLUSION

The sensitivity criterion is widely used in measuring the level of fine-tuning, although it is not reliable. In this paper we investigated the mathematics behind the fine-tuning problems, categories these fine-tuning problems into two types: the first and the second type. Then based on the probability theory, we analyzed the problems existed in the sensitivity criterion, and find three assumptions that widely ignored. By analyzing the high sensitivity problems of Λ_{QCD} and Z-boson mass, we found the reason why the sensitivity criterion failed is because of the dimensional effect, because of the dimensional effect the sensitivity is much larger than the level of real fine-tuning. This effect should be removed when considering fine-tuning problems in renormalization. Based on these analysis, we found the traditional sensitivity criterion can not be used to measure the fine-tuning level for the first type, although it did reflect the fine-tuning level qualitatively for the second type. Based on the probability theory, we proposed a new criterion that can solve all these problems, and verify our new criterion with various fine-tuning problems.

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